Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

4 - 8 Double Fourier Series
Represent f(x,y) by a series (15), where

$$u[x, y, 0] = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \, \text{Sin}\left[\frac{m \pi x}{a}\right] \text{Sin}\left[\frac{n \pi y}{b}\right] = f[x, y]$$

and

5. f(x,y)=y, a=b=1

```
ClearAll["Global`*"]
```

For this type of problem, numbered line (15) is shown above. After a little development, the text presents numbered line (18), p.582, which is the **generalized Euler formula**:

$$B_{mn} = \frac{4}{a\,b} \int_0^b \int_0^a f[x, y] \, \text{Sin}\left[\frac{m \pi x}{a}\right] \text{Sin}\left[\frac{n \pi y}{b}\right] dx \, dy \qquad n, m \to 1, 2, \ldots$$

in the case of this problem,

$$\text{Bmn} = 4 \int_0^1 \int_0^1 y \, \text{Sin}[m \pi x] \, \text{Sin}[n \pi y] \, dx \, dy$$

$$\frac{4 (-1 + \text{Cos}[m \pi]) (n \pi \text{Cos}[n \pi] - \text{Sin}[n \pi])}{m \, n^2 \pi^3}$$

If m is even then Bmn is zero (because **Cos[m π]** would then equal 1), else if m is odd, then

$$\text{Bmno} = \text{Bmn} \,/.\, \{(-1 + \text{Cos}[m \pi]) \to -2, \text{Cos}[n \pi] \to (-1)^{n+1}, \text{Sin}[n \pi] \to 0\}$$

$$-\frac{8 (-1)^{1+n}}{m \, n \, \pi^2}$$

The green cell above matches the text answer for B_{mn}. There is no text answer for u(x,y,0), but it would be the pattern shown above, in cyan, with the restriction that m be odd. The general token B_{1n} does not have a negative sign, which I guess is why $(-1)^{n+1}$ was chosen as the formula for the sign of Cos[n π].

7. f(x,y)=x y, a and b arbitrary

$$\text{Bmn} = \frac{4}{a\,b} \int_0^b \int_0^a x \, y \, \text{Sin}\left[\frac{m \pi x}{a}\right] \text{Sin}\left[\frac{n \pi y}{b}\right] dx \, dy$$

$$\frac{4 \, a \, b \, (m \pi \text{Cos}[m \pi] - \text{Sin}[m \pi]) (n \pi \text{Cos}[n \pi] - \text{Sin}[n \pi])}{m^2 \, n^2 \, \pi^4}$$

The circumstances are not the same as in the last problem. No pattern, even or odd, makes B_{mn} equal zero.

```
Bmno = Bmn /. {Sin[n π] → 0, Sin[m π] → 0}
```

$$\frac{4\,a\,b\,\text{Cos}[m\,\pi]\,\text{Cos}[n\,\pi]}{m\,n\,\pi^2}$$

I hope the above cell would do if required, because I had to cheat by looking at the answer to see the clever device for getting the sign:

```
Bmnf = Bmno /. Cos[m π] Cos[n π] → (-1)^(m+n)
```

$$\frac{4\,(-1)^{m+n}\,a\,b}{m\,n\,\pi^2}$$

The above green cell matches the text answer.

11 - 13 Square Membrane
Find the deflection u(x,y,t) of the square membrane of side π and $c^2 = 1$ for initial velocity 0 and initial deflection

11. 0.1 Sin[2 x] Sin[4 y]

To do this problem I need numbered line (9) on p. 580:

$$\lambda = \lambda_{mn} = c\,\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} \qquad m,\ n \to 1,\ 2,\ 3,\ \ldots$$

and numbered line (14) on p. 582 :

$$u[x,\ y,\ t] = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\text{Bmn Cos}[\lambda_{mn}\,t] + \text{Bast}_{mn}\,\text{Sin}[\lambda_{mn}\,t]\right)\text{Sin}\left[\frac{m\,\pi\,x}{a}\right]\text{Sin}\left[\frac{n\,\pi\,y}{b}\right]$$

and numbered line (18) on p. 582 :

$$B_{mn} = \frac{4}{a\,b}\int_0^b\int_0^a f[x,\ y]\,\text{Sin}\left[\frac{m\,\pi\,x}{a}\right]\text{Sin}\left[\frac{n\,\pi\,y}{b}\right]dx\,dy \qquad m,\ n \to 1,\ 2,\ 3,\ \ldots$$

as well as numbered line (19) on p. 583 :

$$\text{Bast}_{mn} = \frac{4}{a\,b\,\lambda_{mn}}\int_0^b\int_0^a g[x,\ y]\,\text{Sin}\left[\frac{m\,\pi\,x}{a}\right]\text{Sin}\left[\frac{n\,\pi\,y}{a}\right]dx\,dy \qquad m,$$

$n \to 1,\ 2,\ 3\ \ldots$

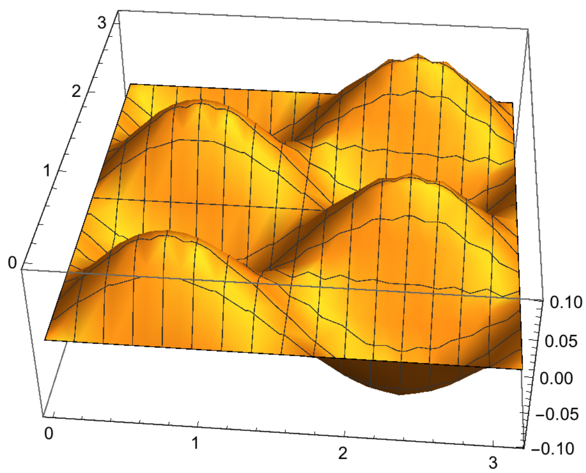
```
In[8]:= ClearAll["Global`*"]
```

The initial displacement is defined. This displacement has the effect of looking very bumpy.

In[9]:=
```
f[x_, y_] = 0.1 Sin[2 x] Sin[4 y]
```
Out[9]=
```
0.1 Sin[2 x] Sin[4 y]
```

The initial displacement is plotted below. The text answer for the solution is 0.1 Cos[$\sqrt{20}\,t$] Sin[2 x] Sin[4 y], a very neat little answer. But giving the square membrane the side length of π throws a monkey wrench into the works, because, using the standard formula, it creates zero terms.

In[11]:=
```
Plot3D[f[x, y], {x, 0, π}, {y, 0, π}, ImageSize → 300]
```
Out[11]=



The Fourier coefficients are computed. To keep everything from vanishing, the square's sides are set to be 3.14159 instead of π.

In[29]:=
```
a[n_, m_] =
  Integrate[Integrate[f[x, y] Sin[m π y/3.14159], {y, 0, 3.14}] Sin[n π x/3.14159],
   {x, 0, 3.14159}];
Grid[Table[a[n, m], {n, 1, 4}, {m, 1, 4}], Frame → All]
```
Out[30]=

1.26134×10^{-13} + 0. ⅈ	-3.14859×10^{-13} + 0. ⅈ	8.07602×10^{-13} + 0. ⅈ	2.77884×10^{-7} + 0. ⅈ
1.11997×10^{-7} + 0. ⅈ	-2.79572×10^{-7} + 0. ⅈ	7.17091×10^{-7} + 0. ⅈ	0.24674 + 0. ⅈ
-2.2704×10^{-13} + 0. ⅈ	5.66744×10^{-13} + 0. ⅈ	-1.45368×10^{-12} + 0. ⅈ	-5.00189×10^{-7} + 0. ⅈ
1.26133×10^{-13} + 0. ⅈ	-3.14858×10^{-13} + 0. ⅈ	8.076×10^{-13} + 0. ⅈ	2.77883×10^{-7} + 0. ⅈ

```
(*Lambda[n_,m_]=(n π)^2 + (m π/2)^2;*)
```

The eigenvalues are computed. Again, the formula is altered in order to hold zero at bay.

In[31]:= Lambda[m_, n_] = $\pi \sqrt{\frac{m^2}{(3.14159)^2} + \frac{n^2}{(3.14159)^2}}$

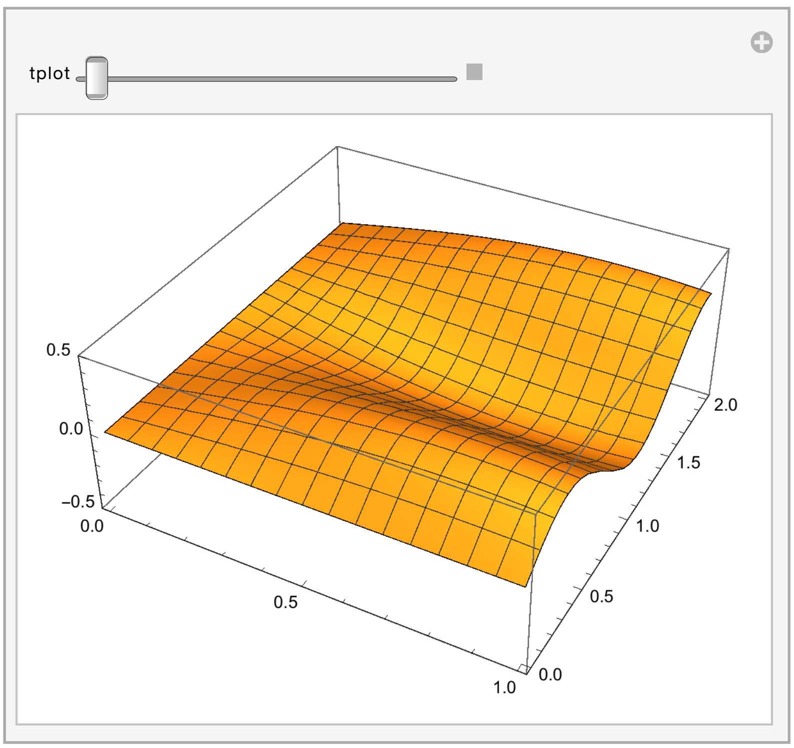
Out[31]= $\sqrt{0.101321\, m^2 + 0.101321\, n^2}\ \pi$

The solution, truncated at N and M, respectively, and with the usual substitution of 3.14159 for π, is given by

```
In[32]:= u[x_, y_, t_, N_, M_] :=
  Sum[Sum[a[n, m] Cos[Lambda[n, m] t] Sin[n π x / 3.14159] Sin[m π y / 3.14159],
    {n, 1, N}], {m, 1, M}]
uplot = u[x, y, t, 4, 4];
```

```
In[34]:= Manipulate[Plot3D[uplot /. t → tplot, {x, 0, 1}, {y, 0, 2},
   PlotRange → {All, All, {-1/2, 1/2}}], {tplot, 0, 5}]
```

Out[34]=



In[35]:= **u[x, y, t, 2, 4]**

Out[35]=

$$(1.26134\times10^{-13} + 0.\,i)\ \mathrm{Cos}[1.41421\,t]\ \mathrm{Sin}[1.\,x]\ \mathrm{Sin}[1.\,y] + (1.11997\times10^{-7} + 0.\,i)\ \mathrm{Cos}[2.23607\,t]\ \mathrm{Sin}[2.\,x]\ \mathrm{Sin}[1.\,y] - (3.14859\times10^{-13} + 0.\,i)\ \mathrm{Cos}[2.23607\,t]\ \mathrm{Sin}[1.\,x]\ \mathrm{Sin}[2.\,y] - (2.79572\times10^{-7} + 0.\,i)\ \mathrm{Cos}[2.82843\,t]\ \mathrm{Sin}[2.\,x]\ \mathrm{Sin}[2.\,y] + (8.07602\times10^{-13} + 0.\,i)\ \mathrm{Cos}[3.16228\,t]\ \mathrm{Sin}[1.\,x]\ \mathrm{Sin}[3.\,y] + (7.17091\times10^{-7} + 0.\,i)\ \mathrm{Cos}[3.60555\,t]\ \mathrm{Sin}[2.\,x]\ \mathrm{Sin}[3.\,y] + (2.77884\times10^{-7} + 0.\,i)\ \mathrm{Cos}[4.12311\,t]\ \mathrm{Sin}[1.\,x]\ \mathrm{Sin}[4.\,y] + (0.24674 + 0.\,i)\ \mathrm{Cos}[4.47214\,t]\ \mathrm{Sin}[2.\,x]\ \mathrm{Sin}[4.\,y]$$

Taking a look at the value of the argument to the first cosine, I find

In[36]:= $(4.47214)^2$

Out[36]= **20.**

Although the above yellow cell is not exactly the text answer, it has the same form. It is seen that the text answer is nearly contained in the last yellow line, except that the leading coefficient differs.

****

The next several cells are an attempt to explore a way to the text answer. However, they are not supported by logic.

Following example 2 on p. 582,

c = 1

1

To test Mathematica's abilities, B_{mn} is calculted two ways:

$$B_{mn} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} .1\ \mathrm{Sin}[2\,x]\ \mathrm{Sin}[4\,y]\ \mathrm{Sin}[m\,x]\ \mathrm{Sin}[n\,y]\,dy\,dx$$

$$\frac{0.324228\ \mathrm{Sin}[m\,\pi]\ \mathrm{Sin}[n\,\pi]}{(-4. + m^2)\ (-16. + n^2)}$$

$$Bmny = \frac{.4}{\pi^2}\int_0^{\pi}\frac{1}{2}\ (\mathrm{Cos}[(n-4)\,y] - \mathrm{Cos}[(n+4)\,y])\,dy$$

$$\frac{0.162114\ \mathrm{Sin}[n\,\pi]}{-16 + n^2}$$

$$\text{Bmnx} = \int_0^{\pi} \frac{1}{2} (\text{Cos}[(2-m)\,x] - \text{Cos}[(2+m)\,x])\, dx$$

$$\frac{2\,\text{Sin}[m\,\pi]}{-4+m^2}$$

bmn = Bmny Bmnx

$$\frac{0.324228\,\text{Sin}[m\,\pi]\,\text{Sin}[n\,\pi]}{(-4+m^2)\,(-16+n^2)}$$

$$\text{bmnc} = \frac{0.3242277876554809\text{`}\,(\text{Cos}[\pi\,(m-n)] - \text{Cos}[\pi\,(m+n)])\,\frac{1}{2}}{(-4+m^2)\,(-16+n^2)}$$

$$\frac{0.162114\,(\text{Cos}[(m-n)\,\pi] - \text{Cos}[(m+n)\,\pi])}{(-4+m^2)\,(-16+n^2)}$$

bmnc makes use of a trig identity that I hoped would help out with the usability of bm.

$$\lambda_{mn} = \text{Simplify}\left[c\,\pi\,\sqrt{\frac{m^2}{\pi^2} + \frac{n^2}{\pi^2}}\,\right]$$

$$\sqrt{m^2+n^2}$$

$$\text{uxyt2} = \text{Sum}\left[\text{Sum}\left[\text{bmnc}\,\left(\frac{1}{2}\,(\text{Cos}[m\,x\ -\ n\,y] - \text{Cos}[m\,x\ +\ n\,y])\right)\,\text{Cos}[\lambda_{mn}\,t],\ \{m,\ 1,\ \infty,\ 2\}\right],\ \{n,\ 1,\ \infty,\ 2\}\right]$$

$$\text{Sum}\left[\text{Sum}\left[\left(0.0810569\,(\text{Cos}[(m-n)\,\pi] - \text{Cos}[(m+n)\,\pi])\,\text{Cos}\left[\sqrt{m^2+n^2}\,\,t\right]\,(\text{Cos}[m\,x - n\,y] - \text{Cos}[m\,x + n\,y])\right) \Big/ \left((-4+m^2)\,(-16+n^2)\right),\ \{m,\ 1,\ \infty,\ 2\}\right],\ \{n,\ 1,\ \infty,\ 2\}\right]$$

The trig identity is used again. But there are still problems.

I was not able to get the text answer, which looks very nice. The yellow cell above, which was assembled using the rules as best as I was able to apply them, looks very messy.

$$\text{u}[x\_,\ y\_,\ t\_,\ m\_,\ n\_,\ j\_,\ k\_] := \text{Sum}\left[\text{Sum}\left[\frac{1}{(-4+m^2)\,(-16+n^2)}\,0.08105694691387022\text{`}\,(\text{Cos}[(m-n)\,\pi] - \text{Cos}[(m+n)\,\pi])\,\text{Cos}\left[\sqrt{m^2+n^2}\,\,t\right]\,(\text{Cos}[m\,x - n\,y] - \text{Cos}[m\,x + n\,y]),\ \{m,\ 1,\ \infty,\ 2\}\right],\ \{n,\ 1,\ \infty,\ 2\}\right]$$

It’s not blowing up at the moment, but I have little hope of verifying it.

```
u[x, y, t, m, n, 1, 1]
```

$$\text{Sum}\left[\text{Sum}\left[\left(0.0810569\,(\text{Cos}[(m-n)\,\pi]-\text{Cos}[(m+n)\,\pi])\,\text{Cos}\left[\sqrt{m^2+n^2}\;t\right]\,(\text{Cos}[m\,x-n\,y]-\text{Cos}[m\,x+n\,y])\right)\Big/\left(\left(-4+m^2\right)\left(-16+n^2\right)\right),\ \{m,1,\infty,2\}\right],\ \{n,1,\infty,2\}\right]$$

****

```
partialuo[j_, k_, x_, y_, t_] :=
 Sum[0.1 Sin[m x] Sin[n y] Cos[Sqrt[m^2 + n^2] t], {m, j, j, 2}, {n, k, k, 2}]
part1 = partialuo[2, 4, x, y, t]
```

$0.1\,\text{Cos}\left[2\sqrt{5}\;t\right]\,\text{Sin}[2\,x]\,\text{Sin}[4\,y]$

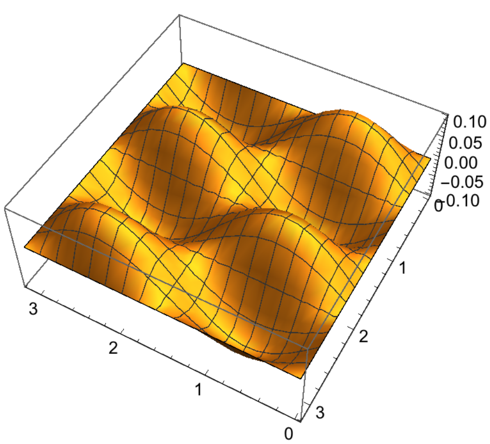
Just playing around, trying to get the text answer, I came up with the above yellow cell. The below yellow cell also gets there. No justification for this entertainment unfortunately.

```
f[x_, y_] = 0.1 Sin[2 x] Sin[4 y]
0.1 Sin[2 x] Sin[4 y]

partialuo2[j_, k_, x_, y_, t_] :=
 Sum[f[x, y] Cos[Sqrt[m^2 + n^2] t], {m, j, j, 2}, {n, k, k, 2}]
part2 = partialuo[2, 4, x, y, t]
```

$0.1\,\text{Cos}\left[2\sqrt{5}\;t\right]\,\text{Sin}[2\,x]\,\text{Sin}[4\,y]$

```
uti = 0.1 Sin[2 x] Sin[4 y];
Plot3D[Evaluate[uti], {x, 0, Pi}, {y, 0, Pi},
 PlotPoints → {20, 20}, ViewPoint → {-1.5, 3, 0.5}]
```



13. 0.1 x y(π - x)(π - y)

```
ClearAll["Global`*"]
```

$$c = 1;\ \lambda_{mn} = \text{Simplify}\left[c\,\pi\sqrt{\frac{m^2}{\pi^2} + \frac{n^2}{\pi^2}}\right]$$

$$\sqrt{m^2 + n^2}$$

```
f[x_, y_] = 0.1 x y (π - x) (π - y)
0.1 (π - x) x (π - y) y
```

The expression in the cell below is not complete, because I have pulled out a factor of $.4/\pi^2$ to save for later.

$$\text{Bmnl} = \int_0^\pi\int_0^\pi x\,y\,(\pi - x)\,(\pi - y)\,\text{Sin}[m\,x]\,\text{Sin}[n\,y]\,dx\,dy$$

$$\frac{(-2 + 2\,\text{Cos}[m\,\pi] + m\,\pi\,\text{Sin}[m\,\pi])\,(-2 + 2\,\text{Cos}[n\,\pi] + n\,\pi\,\text{Sin}[n\,\pi])}{m^3\,n^3}$$

Now as for the cell above: I want to save the denominator, and to evaluate the numerator:

```
Sum[(-2 + 2 Cos[m π] + m π Sin[m π]) (-2 + 2 Cos[n π] + n π Sin[n π]),
 {m, 1, 1, 2}, {n, 1, 1, 2}]
16

6.4 / .4
16.
```

Combining the operations above means that I have a total leading factor now of $\frac{6.4}{\pi^2}$, and all that is left of Bmnl is $\frac{1}{m^3\,n^3}$.

$$\text{outeq} = \text{Simplify}\left[\text{partialu}[j\_, k\_, x\_, y\_, t\_] := \sum_{n=1}^{k}\sum_{m=1}^{j}\text{Bmn}\,\text{Sin}[m\,x]\,\text{Sin}[n\,y]\,\text{Cos}\left[\sqrt{m^2 + n^2}\;t\right],\ \text{Assumptions} \rightarrow \{m, n \in \text{OddQ}\}\right]$$

$$\text{outeq1} = \text{Simplify}\left[\text{partialu}[j\_, k\_, x\_, y\_, t\_] := \frac{6.4}{\pi^2}\sum_{n=1}^{k}\sum_{m=1}^{j}\frac{1}{m^3\,n^3}\,\text{Sin}[m\,x]\,\text{Sin}[n\,y]\,\text{Cos}\left[\sqrt{m^2 + n^2}\;t\right],\ \text{Assumptions} \rightarrow \{m, n \in \text{OddQ}\}\right]$$

The above green cell matches the answer of the text.

14 - 19 Rectangular Membrane

17. Find eigenvalues of the rectangular membrane of sides a = 2 and b = 1 to which there correspond two or more different (independent) eigenfunctions.

```
ClearAll["Global`*"]
```

$\lambda_{mn} = c\,\pi \sqrt{\frac{m^2}{4} + \frac{n^2}{1}}$

$c \sqrt{\frac{m^2}{4} + n^2}\ \pi$

$eig[m\_, n\_] = \frac{m^2}{4} + n^2$

$\frac{m^2}{4} + n^2$

$Solve\left[\frac{m^2}{4} == n^2, \{m, n\}\right]$

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

$\left\{\left\{n \to -\frac{m}{2}\right\}, \left\{n \to \frac{m}{2}\right\}\right\}$

```
Table[eig[m, n], {m, {4, 16}}, {n, {16, 14}}]
{{260, 200}, {320, 260}}
```

```
eig[10, 5]
50
```

```
Table[eig[m, n], {m, {2, 4, 6, 8}}, {n, {1, 2, 3, 4}}]
{{2, 5, 10, 17}, {5, 8, 13, 20}, {10, 13, 18, 25}, {17, 20, 25, 32}}
```

So for example $\lambda_{8,3} = \lambda_{6,4} = c\sqrt{25}\ \pi$. These are much smaller m,n than the text uses in its answer, but if I understand correctly, they are okay.

19. Deflection. Find the deflection of the membrane of sides a and b with $c^2 = 1$ for the initial deflection
$f(x, y) = Sin\left[\frac{6\pi x}{a}\right] Sin\left[\frac{2\pi y}{b}\right]$ and initial velocity 0.

```
ClearAll["Global`*"]
```

$c = 1;\ \lambda_{mn} = Simplify\left[c\,\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}\right]$

$\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}\ \pi$

$$f[x\_, y\_] = \text{Sin}\left[\frac{6\,\pi\,x}{a}\right]\,\text{Sin}\left[\frac{2\,\pi\,y}{b}\right]$$

$$\text{Sin}\left[\frac{6\,\pi\,x}{a}\right]\,\text{Sin}\left[\frac{2\,\pi\,y}{b}\right]$$

$$B_{mn} = \frac{4}{a\,b}\int_0^b\int_0^a f[x, y]\,\text{Sin}\left[\frac{m\,\pi\,x}{a}\right]\,\text{Sin}\left[\frac{n\,\pi\,y}{b}\right]\,dx\,dy$$

$$\frac{48\,\text{Sin}[m\,\pi]\,\text{Sin}[n\,\pi]}{\left(-36 + m^2\right)\,\left(-4 + n^2\right)\,\pi^2}$$

$$u[x\_, y\_, t\_, j\_, k\_] = \sum_{m=1}^{j}\sum_{n=1}^{k}\left(B_{mn}\,\text{Cos}[\lambda_{mn}\,t]\right)\,\text{Sin}\left[\frac{m\,\pi\,x}{a}\right]\,\text{Sin}\left[\frac{n\,\pi\,y}{b}\right]$$

0

This problem has a defect similar to the one found in problem 11, a persistant zero, and I have not figured out how to avoid it.